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Обучение способам составления задач как основа развития профессиональной компетентности будущих учителей математики

На основе анализа педагогического опыта авторами показано, что для эффективного усвоения метода решения задачи, необходимо овладение учащимися способами создания задач, решаемых этим методом. Помимо этого система математического образования в школе требует от будущих учителей владения способами составления задач, что дает возможность формулировки большого и разнообразного круга задач, позволяющих учителю реализовывать различные образовательные цели.

В исследовании проведен анализ способов составления задач, выделены наиболее эффективные, по мнению авторов, для подготовки будущих учителей математики. Подробно рассмотрены три из выделенных способа: введение параметра при решении задачи и дальнейшая его конкретизация при составлении класса задач с прогнозируемыми свойствами решений; составление задач с заданным методом решения в общем виде; формулировка вопросов к так называемой открытой задаче и способы составления открытых задач.

Нами представлена методика обучения студентов выделенным способам составления математических задач при изучении различных дисциплин. Овладение будущими учителями математики этими умениями позволяет осуществлять необходимую многовариантную тестовую проверку знаний учащихся по изучаемой теме, с использованием электронных средств и интернет-ресурсов, лишая учащихся возможности найти ответ в интернете.

Проведенное экспериментальное исследование позволило сделать вывод: обучение будущих учителей математики выделенным способам составления математических задач формирует у них комплекс универсальных и профессиональных компетенций, создавая основу формирования и развития профессиональной компетентности.

Ключевые слова: методика высшей школы, содержание образования студентов педагогических специальностей, компетенции, задачи по математике, методы решения задач

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Teaching ways of compiling tasks as the fundamentals of the development of professional competence of future mathematics teachers

Based on the analysis of pedagogical experience, the authors showed that in order to master the method of solving a problem effectively, students need to master the methods of creating problems solved by this method. In addition, the system of mathematics education at school requires future teachers to master the methods of compiling tasks, which makes it possible to formulate a large and diverse range of tasks that enable the teacher to realize various educational goals.

The research analyzed the ways of compiling the problems; according to the authors, the most effective ones for the training future mathematics teachers are sorted out. Three of the selected ways are discussed in detail: the bringing in a parameter when solving a problem and its further specification when creating a set of problems with predictable properties of solutions; compiling tasks with a given solution method in general terms; the formulation of questions to the so-called open problem and the ways of composing open problems.

We have presents the methods of teaching students the selected ways of compiling mathematical problems in the study of various disciplines. Future teachers of mathematics mastering these skills allows one to perform the necessary multivariate test examination of students' knowledge on the topic being studied, using electronic means and Internet resources, preventing students from finding the answer on the Internet.

The experimental research made it possible to conclude that the training future mathematics teachers to apply these methods of compiling mathematical problems practically forms a complex of universal and professional competences, creating the basis for the formation and development of their professional competence.

Keywords: higher school methodology, education of students of pedagogical specialties, competences, math problems, methods of solving problems

For Reference:
Introduction

The school teacher is the main figure involved in the educational process. Conducting a survey among teachers of the city of Magnitogorsk, we made the following conclusions:

1. Even using the resources of the Internet, it is difficult to make multiple verification work of the same level of complexity. Taking the task from the electronic resources presented there, we have no guarantee that the student will not use the answer obtained in the same way.

2. Many teachers have difficulty in doing project research work with students. There appear difficulties in formulating applied mathematical problems, in selecting necessary and sufficient conditions for their solution.

3. There are serious difficulties in solving academic or research mathematical problems requiring the synthesis of various approaches in their implementation.

The opportunities to overcome these problems based on the use of cloud computing were investigated in this research [1]. However, the experience exchange between teachers, unfortunately, does not increase the number of mathematical problems, the open solution of which is not available to the student when using the Internet. And for the development of electronic learning means they need a large number of tasks of the same type or one level of complexity [2-4].

In the study, we wanted to pay special attention to the fact that whatever opportunities Internet technologies provide us, the most important of the future teacher's competencies are competencies that allow them to develop the skills of independent task writing. We corroborated the need to train students, future mathematics teachers, methods of compiling problems. Obviously, it is difficult to evaluate the effectiveness of this approach experimentally if during the training of mathematics teachers at university, attention was paid not only to methods for solving mathematical problems, but also to methods for compiling them. It is no less difficult to estimate the time and effort savings of a teacher in selecting tasks, in compiling multivariate tests. It is clear that by teaching the future teacher to formulate tasks, we will greatly facilitate their life. Therefore, in this article, the aim was to analyze the ways of compiling tasks, select the most suitable of them for teaching mathematics, describe some of them used in the course of the experiment. Based on the analysis of the experiment, we affirm that for the most effective assimilation of the method of solving a problem, it is necessary to master the ways of creating problems solved by this method. This conclusion is valid not only for future mathematics teachers, but also for students from other areas, which confirms our hypothesis once again.

Methods

One of the spheres of preparing students for the Unified State Exam is to develop the skills of passing the test examination of students' knowledge, which requires the teacher to create extensive array of the same type of tasks, allowing to form different versions of tests of equal complexity [18]. The power of each created set of tasks correlates with the number of variants of one level of complexity that can be offered to students. Testing students' knowledge allows to fulfil the diagnostic analysis of the level of material learning. And also it
can be carried out for educational and training purposes. But how can one make up at least the necessary minimum of tasks without burdening the teacher with multiple checks on the work of the students? How can one help the teacher to implement these ideas? To do this, it is necessary to equip the teacher with the knowledge of task-writing methods. To support our thesis, there is also an insufficient number of systematized collections of problems, and the availability for a student of solving a non-authoring task on the Internet [5; 6; 19].

The second, no less important, argument in favor of the importance of mastering the skills of creating tasks is reduced to a deeper understanding of the method for solving problems of a given type. Practicing the skill of solving problems of this set will not be complete if the student does not understand the specifics of the formulation of the problems solved by this method. The ability to make out a problem with a predictable method of solution indicates a deep understanding and insight into the method of solving problems of this type.

Corroborating the need for students of pedagogical specialties to learn how to compile tasks, you can use the complementarity principle proposed by G.G. Granatov. Of all the tenets of this principle, we will use the fact that in any pedagogical phenomenon there must exist a pair of complementary elements. For the most effective assimilation of the method of solving a problem, it is necessary to form the skills of creating problems solved by this method. The third argument of the necessity of teaching methods of compiling problems is the development of mathematical creativity and understanding the possibilities of mathematics when solving applied problems compiled by students [7].

Such wise, to achieve this goal, it is suggested using a methodology for teaching students to compile mathematical problems.

We emphasize the following methods of forming new tasks that teachers must master:
1. Making tasks with a given solution method in general terms.
2. Finding subtasks of the direct problem, their formulation and solution.
3. Compilation of applied problems using the selection of the necessary and sufficient conditions of the analyzed situation, elimination of unnecessary requirements, the addition of data on the incomplete situation [8; 9].
4. Creating tasks of the same type with variable numerical data.
5. Introducing a parameter when solving a problem and further specifying it when creating a set of problems with predictable properties of solutions.
6. Making a task according to the general scheme of a brief record of the condition [10].
7. Compilation of inverse problems.
8. The formulation of questions for the so-called open problem, methods for composing open problems [9].

This method has been experimentally tested in the process of teaching students of different specialties at Nosov Magnitogorsk State Technical University.

The hypothesis of our research is the proposition that training future mathematics teachers to use the described ways of compiling mathematical problems forms in them a set of universal and professional competences, and at the same time creates the basis for the formation and development of their professional competence.

**Results**

Let us dwell on teaching students some ways to make new tasks. Let us consider the method of introducing a parameter when solving a problem and further specifying it to
obtain a set of problems with a predictable property of solutions.

Great opportunities in the compilation of tasks provide us with tasks with a parameter. For example, let us consider the following problem: Solve the equation \( F(x,a) = 0 \), where \( a \) is a parameter. It is clear that for each value of the parameter, we obtain the formula of solutions or prove that the equations have no solutions. Having solved this equation, we will have great opportunities in compiling new problems. For this, it is enough to substitute a specific number instead of the parameter. And depending on which range this number falls into, we get the solution formula or we will have an equation that has no solutions. Thus, having taught students to solve problems with parameters, we give them an effective method of making tasks that do not contain a parameter and satisfy the properties necessary at this stage of training. Let us note that the compilation of tasks at this level can also be entrusted to electronic means [2; 3; 11].

Let us consider this approach in the simplest example. When working out with students the method of teaching students how to solve quadratic equations, one or several parameters can be entered into a quadratic equation to obtain multivariate answers. Solving it, and then, using the specification of the parameters, we get many of the same type of tasks with given restrictions on the solutions.

For example, it is suggested solving a quadratic equation with the parameter \((x+a)+b^2=0\).

After solving the equation above we get the following answer:
- if \(|a|<2|b|\), then the equation has no solution;
- if \(|a|=2|b|\), then the equation has one \(x = -a/2\);
- if \(|a|>2|b|\), then the equation has two solutions

\[
\chi = \frac{-a \pm \sqrt{a^2-4b^2}}{2}
\]

The answer allows us to make a range of quadratic equations of the given form, with two solutions, a unique solution and no solutions. To do this, it is necessary to take specific values of the parameters that satisfy the corresponding limitations in the answer. In addition, we have a general solution formula that allows us to verify the correctness of student responses quickly.

Geometry gives a lot of opportunities for the formation of skills to create tasks. Due to the diversity of the geometric material, the logic of the structure and the solution of geometric problems, we are constantly faced with the need to find the subtasks of the direct problem, their formulation and solution, with the formulation of applied problems with the analysis of the selection of necessary and sufficient conditions of the situation under consideration [12-14]. Great opportunities for the formulation of new problems are provided by geometrical tasks for construction with a pair of compasses and a ruler, especially at the stage of analyzing a solved problem. So, based on the analysis of the construction problem, we can formulate problems with given properties of solutions.

For example, when solving a classical problem of building a triangle on three sides \(a\), \(b\) and \(c\) \((a \geq b \geq c)\), according to the results of the study, we obtain the inequality \(a<b+c\), for which the problem has a solution. After that, students should formulate this task so that it:
1. would have no solution,
2. would have the only solution,
3. would have two solutions,
4. would have four solutions.

As a result, we received such variants of the problem.

Build a triangle:
1. on three sides \(a, b\) and \(c (a\geq b+c);\)
2. on three sides \(a, b\) and \(c (a<b+c);\)
3. a) with apexes at points \(A\) and \(B\) and sides \(a\) and \(b (a = b);\) b) with apexes at points \(B\) and \(C\) and sides \(1\) and \(b (c = b\) and \(BC < c + b);\)
4. with apexes at points \(A\) and \(1\) and sides \(a\) and \(b (a > b).\)

In brackets there are the restrictions guided by which the teacher sets the segments for obtaining the required number of solutions.

A great source for composing new tasks is open problems. Most school tasks have a well-defined condition and the only answer. These are tasks like “find, if you know ...” and “prove that ...”. Such tasks are called closed. An open task does not have a clear unambiguous condition, it may lack data or, on the contrary, it may be redundant. Such a task, in contrast to the closed one, has at least two mutually exclusive answers.

Any open problem can be reformulated by making it closed. Discussing with students how to transform open problems into closed ones, we identified several ways. Depending on which part of the task, the condition or task, makes it open, you can move in the following directions:

1. If there is not enough data in the task, then it can be defined by various conditions.
2. If there is not enough data in the task, then you can change the task.
3. If the task is redefined, then some data can be discarded.
4. If the task is redefined, then the task can be changed.

Students were offered the following condition: “Three segments are given, the lengths of which are referred to as 3: 4: 5. Build a triangle so that... .Consider all possible cases. How many solutions does the problem have for each set of elements? ”

The result of the work was the set of following tasks compiled by students:

a) These segments were his sides and median, emerging from the same apex.

b) Two of these segments were the median and the side coming out of one apex, and the third - the side to which the median was drawn.

c) These segments were its sides and height, emerging from the same apex.

d) Two of these segments were the height and the side coming out of one apex, and the third – the side to which the height was drawn.

A special place from the standpoint of didactics is occupied by the method of composing problems with a given solution method in a general form [15], which was also tested in our methodology.

The solution of the research tasks put forward the necessity of conducting an experiment that allows analyzing the conditionality of the depth of learning of the material under study, the mastery of the methods and the generated skills of task preparation. The purpose of this study is to analyze the level of the formed competencies of future mathematics teachers when teaching them how to compose tasks. To achieve this aim, the following research tasks were solved:

1. A base for the experiment was created. Nosov Magnitogorsk State Technical University (NMSTU).

2. The group of the experiment participants was chosen. The participants included 44 second-year students who study at the “Pedagogical Education” department of the “Mathematics” and “Applied Mathematics and Informatics” profiles. All students are trained in full-time education.

Students were divided into three groups. One group (group A) are students of pedagogical specialty, two groups (groups B and C) are students of the specialty "Applied Mathematics
and Computer Science”. When studying the topic “Application of the function properties when solving equations and inequalities” in the course “Elementary Mathematics” in groups A and B, in addition to studying methods of solving problems, attention was paid to learning composing them. When conducting classes, the ideas from the work were used [16]. Groups A and B were offered tasks that were later used to conduct a test on this topic in the three study groups. The tasks of group B were used in group A, and vice versa.

So, for instance, after updating the method of using the monotonicity of functions when solving equations and inequalities and studying the proper theorem [16], the students were asked to formulate an algorithm for solving equations and inequalities based on the applying of this theorem.

Only after practicing the solution method do students think up an algorithm that allows them to compile tasks solved by this method:

1. Set a monotonic function on the set $X \ y=f(u)$.
2. Find the functions $y = \alpha (x)$ and $y = \beta (x)$.
3. Present the equation (inequality) in the form $f(\alpha(x))=f(\beta(x))$.
4. Formulate the condition of the problem, complicating the necessary equation (inequality): $f(\alpha(x))=f(\beta(x)) (f(\alpha(x))>f(\beta(x)))$, distributing some of the terms in different parts and thereby hiding the general form of the origin function $y=f(u)$.
5. Solve the problem for verification by applying the previously formulated algorithm.

Then we offer students to implement the presented algorithm with given functions $\alpha(x)$, $\beta(x)$, $f(x)$.

The question how to choose a monotonous function arises. To do this, you can use elementary monotone functions, or based on the properties of the sum, difference, and the product of monotonic functions, you can build new ones.

When studying this topic, students compiled multi-level tasks to be solved by this method:

For all the groups:

1. $\cos^5 2x + 6\cos^3 2x = \cos^5 x + 6\cos^3 x$.
2. $\sqrt{2x-1} - \frac{\sqrt{4x+1}}{2} \leq \sqrt{4x+1} - \frac{\sqrt{2x-1}}{2}$.
3. $\sin(2\tan x) - 3\tan x = \sin(2\cot x) - 3\cot x$.

The proper functions are easily determined in these equations and inequalities. The monotonicity of the main function follows from the characteristics of monotone and elementary functions.

Groups A and B.

1. $\sin(x^2 + 1) + (x^2 + 1)^3 + x^2 + 1 + \sin 2x + 8x^3 + 2x = 0$.
2. $\arccot (\sin 2x) - \arccos (\sin 2x) = \arccot (\cos x) - \arccos (\cos x)$.
3. $-2\left(5 - \sqrt{x^2 - 1}\right)^2 - 7\left(5 - \sqrt{x^2 - 1}\right)^3 + \sqrt{x^2 - 1} - 5 \geq 2x^5 + 7x^3 - x$.

Group C.

1. $7\tan x - \cos(2\tan x) \geq 14\sqrt{3} \tan^2 x - \cos(2\sqrt{3} \tan^2 x)$.
2. $\sin \left(9x^2 + \frac{4}{x^2} - 7\right) + \sin \left(3x - \frac{2}{x} - 7\right) = 18x^2 + \frac{8}{x^2} - \frac{4}{x} + 6x - 28$.
3. $\sqrt{1 - x^2} + \arccos(x^2 - 1) \geq \sqrt{1 - 2x} + \arccos(2x - 1)$. 
These tasks are distinguished by complexity in the selection of the function and essential constraints on the domain of the function, and therefore on the values of the functions.

The formation of competencies that contribute to the development of skills for the preparation of tasks should be given constant attention. Thus, the roundedness of functions of different kinds was emphasized while studying the topic “Range of Function”. After studying methods of solving problems related to the values of functions [16], special attention was paid to the algorithm for solving such problems. Having formulated it, having worked on practical tasks, the students asked the question: how can one create tasks for solving which you can use a similar algorithm? That is, having solved the “direct” problem - the direct solution of an equation or inequality, we move on to the “inverse» problem: how to make up a task. The algorithm for solving the "direct" problem helps us to create an algorithm for solving the "inverse" ones. In addition, it is necessary to find answers to the questions: how can one create a function with a given range of values, which functions should be combined in the condition, how can one “hide” the original functions? Having answered these questions, students were able to formulate theorems presented in [16], for the case if the left-hand side is the sum of non-negative functions.

Having learned how to compile problems solved by the suggested method, students were able to formulate theorems, the use of which does not cause difficulties even in the secondary school. Answering these questions arising while composing the problem, students have a deep insight into the essence of the method of its direct solution.

When studying the topic “Applying the properties of functions when solving equations and inequalities”, each direct solution of a series of problems using the theorems under study was supported by an appropriate algorithm for composing problems solved by this method. The feedback allowed students to increase their creative approach in performing tasks, increased their motivation, and a genuine interest appeared: “But who can solve my problem?”, “How can I create tasks of different complexity levels?”. Students had their own collections of tasks, some of which were used to write final qualifying works. The student were so fascinated by this way of making up tasks and the forming new statements that they actually came to the formulating of the new statements themselves. Thus, the students formulated the principle which in mathematics is called the “mini-max principle” themselves, and the algorithms for solving problems by this method were made. After sorting out their main components, the algorithms for the formulation of problems solved using this principle were made.

During the experiment, the students made up a lot of non-trivial tasks that were simultaneously tested and adopted by teachers who attended advanced training courses, and were also used by students in teaching practice.

Let us imagine some of the tasks created by students. These tasks were later used in the final work.

Solve equations and inequalities (see below).

Despite the fact that in the process of studying this topic in groups A and B, it was necessary to pay great attention to the study of methods of compiling tasks, the learning process proceeded more intensively than in Group C, where these methods were not considered. The students had a genuine interest in studying the material, there was some excitement: Who will compile more tasks? Who can solve my problem? The question: Where do the tasks come from? gradually disappeared.

The experience of compiling tasks turned out to be very useful, especially for students who are future teachers. It allowed them to learn how to set and solve problems related
to their future professional activity. Taking into account the importance of the competence approach in the training of future teachers [17], during the experiment we were repeatedly convinced of the conclusion: teaching students different ways of writing mathematical tasks forms a set of universal and professional competences that will contribute to the formation and development of their professional competence later on.

\[
\begin{align*}
x^4 + 5 \cdot 4^x + 4x^2 \cdot 2^x - 2 \cdot 2^x + 1 &= 0, \\
1 - \sqrt{1 - x^2 - x^4} + \log_2 (1 + x^2) &= 0, \\
x - 3 + |\log_{0.7} (x^2 - 4x + 4)| &= 0, \\
\sqrt{x^2 - 7x + 12 + \log_2 (x^2 - 4x + 1)} &\leq 0, \\
(x^2 - 5x + 6)^2 + \log (x^2 - 4x + 5) &\leq 0, \\
\frac{\sqrt{x^4 + 5x^3 + 64} + \arcsin^2 (x^2 + 4x)}{x^2 - x + 1}, \\
|\log (x^2 + 2x + 2) + 5 - 4 + 2x + x^2| &\leq 0, \\
4x^2 + 4x + 17 &= \frac{12}{x^2 - x + 1}, \\
\cos^2 (x \sin x) = 1 + |\log_2 (x^2 - x + 1)|, \\
|\log (x^2 + 2x + 2) + 5| &\leq 4 - 2x - x^2, \\
|\log (x - 2)| + 1 &\leq -\cos \pi x; \\
\frac{\cos^2 x}{\sin^8 x} + \frac{\sin x}{\cos^2 x} &= 2 \cos^2 \frac{\pi^2}{4} - x^2.
\end{align*}
\]

Discussion

The final stage of the experiment was a final test on the topic “Application of the function properties when solving equations and inequalities”. The test included tasks of two levels. Level 1: these are tasks created by the teacher for all three groups. Level 2: these are separate tasks proposed for solving groups A and B, and tasks for group B. The tasks of the second level were formulated during the experiment by the students themselves after studying the ways of composing them. A more detailed description of the division of tasks and tasks themselves are presented above.

The results of test are contained in Table 1.

<table>
<thead>
<tr>
<th>Group</th>
<th>Total students in the group</th>
<th>the number of students who, in general, coped with the tasks of level 1</th>
<th>the number of students who, in general, coped with the tasks of the level 2</th>
<th>% of students who received an excellent and a good marks</th>
<th>% of students who received a bad mark</th>
<th>% of students who passed a test on the topic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group A</td>
<td>15</td>
<td>15</td>
<td>11</td>
<td>73%</td>
<td>27%</td>
<td>100%</td>
</tr>
<tr>
<td>Group B</td>
<td>14</td>
<td>14</td>
<td>12</td>
<td>86%</td>
<td>14%</td>
<td>100%</td>
</tr>
<tr>
<td>Group C</td>
<td>15</td>
<td>12</td>
<td>2</td>
<td>13%</td>
<td>67%</td>
<td>80%</td>
</tr>
</tbody>
</table>

As can be seen from the data in Table 1, the percentage of students in groups where methods of compiling tasks that qualitatively solved a test were studied is several times higher than the percentage of the same students, group B, where it was not paid enough attention to methods and skills in composing tasks. Further observation of the learning outcomes of the selected groups of students showed that knowledge of the methods and
formed skills of creating task allowed them later to write coursework and final qualifying works successfully. The students, future teachers of mathematics (group A), successfully coped with the selection and formulation of new problems, using the methods of their compositing studied earlier.

As a result of the experiment, the following points were formulated:
1. In order to practice the skill of solving problems of this type fully, it is necessary to show students the specifics of how the tasks are solved by this method. The ability to create a problem with a predictable method of solution indicates a deep insight into the method of solving problems of this type.
2. Teaching compiling problems contributes to the development of mathematical creativity, understanding the possibilities of mathematics in solving applied, production problems composed by students.
3. Formation of skills for compiling tasks, knowledge of ways of their compilation allows successfully to conduct research aimed at improving methods for solving standard professional tasks based on information and bibliographic culture using information and communication technologies and taking into account the basic requirements of information security.
4. The ability to formulate tasks allows one to master and apply those methodological technologies that future teachers need for individual work with a different contingent of students, both with gifted children and with children with disabilities.

Conclusion

The system of modern mathematics education at school requires teachers to master the methods of drawing up tasks, which makes it possible to formulate a large and diverse range of tasks that enable the teacher to realize various educational goals.

The results of the experiment fully confirm the hypothesis of our study: the training of future mathematics teachers to selected methods of compiling mathematical problems forms in them a complex of universal and professional competences, creating the basis for the formation and development of professional competence.

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